

Assisted coherence distillation of certain mixed states

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Abstract In the task of assisted coherence distillation via the set of operations X , where X is either local incoherent operations and classical communication (LICC), local quantum-incoherent operations and classical communication (LQICC), separable incoherent operations (SI), or separable quantum-incoherent operations (SQI), two parties, namely Alice and Bob, share many copies of a bipartite joint state. The aim of the process is to generate the maximal possible coherence on the subsystem of Bob. In this paper, we investigate the assisted coherence distillation of some special mixed states, the states with vanished basis-dependent discord and Werner states. We show that all the four sets of operations are equivalent for assisted coherence distillation, whenever Alice and Bob share one of those mixed quantum states. Moreover, we prove that the assisted coherence distillation of the former can reach the upper bound, namely QI relative entropy, while that of the latter can not. Meanwhile, we also present a sufficient condition such that the assistance of Alice via the set of operations X can not help Bob improve his distillable coherence, and this condition is that the state shared by Alice and Bob has vanished basis-dependent discord.

Keywords quantum coherence · coherence distillation · QI relative entropy

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1 Introduction

Quantum coherence, another embodiment of the superposition principle of states, is essential for many distinctive and captivating characteristics of quantum systems [1-3]. The quantification of quantum coherence was recently proposed by Baumgratz *et al.* [4]. Under this framework, many new coherence quantifiers have been presented since then [5-8]. Meanwhile, various properties of quantum coherence have been investigated such as the connections between quantum coherence and quantum correlations [9-14], the distillation of coherence [15-17], the dynamics under noisy evolution of quantum coherence [18,19], among others. The role of coherence in biological system [20], and thermodynamics [21] has also been explored.

In the framework of coherence given by Baumgratz *et al.* [4], let $\{|i\rangle\}$ be some fixed reference basis (incoherent basis) in the finite dimensional Hilbert space, and a state is said to be incoherent if it is diagonal in this basis, being of the form $\sum_i p_i |i\rangle\langle i|$. A quantum operation is identified by a set of Kraus operators $\{K_l\}$ satisfying $\sum_l K_l^\dagger K_l = I$. Specially, a quantum operation is called incoherent if it can be written in the form $\Lambda(\rho) = K_l \rho K_l^\dagger$, with incoherent Kraus operators K_l , i.e., $K_l|m\rangle \sim |n\rangle$, where $|m\rangle$ and $|n\rangle$ are elements of the fixed reference basis. As a quantifier of quantum coherence, we will use the relative entropy of coherence, initially defined as $C_{re}(\rho) = \min_{\sigma \in \mathcal{I}} S(\rho||\sigma)$, where $S(\rho||\sigma) = \text{Tr}(\rho \log_2 \rho) - \text{Tr}(\rho \log_2 \sigma)$ is the quantum relative entropy [22] and the minimization is taken over the set of all incoherent states \mathcal{I} . Crucially, the relative entropy of coherence can be evaluate exactly: $C_{re}(\rho) = S(\Pi(\rho)) - S(\rho)$, where $S(\rho) = -\text{Tr}(\rho \log_2(\rho))$ is the von Neumann entropy [22] and $\Pi(\rho) = \sum_i |i\rangle\langle i| \rho |i\rangle\langle i|$ denotes the von Neumann measurement (dephasing operation) of ρ with respect to the fixed reference basis. An important progress within the resource theory of coherence is the operational theory of coherence, which is introduced by Winter and Yang [15]. Particularly, they presented the distillable coherence for any state ρ , i.e., $C_d(\rho) = C_{re}(\rho)$, where the distillable coherence $C_d(\rho)$ is defined as the maximal rate for extracting maximally coherent single-qubit states, $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, from a given state ρ via incoherence operations.

For a bipartite system AB shared by Alice and Bob, its reference basis is usually assumed to be the tensor product basis of local bases [6], i.e., $\{|ij\rangle^{AB}\}$, where $\{|i\rangle^A\}$ and $\{|j\rangle^B\}$ are the fixed reference bases of subsystems A and B , respectively. In the task of assisted coherence distillation [16, 17], two parties, namely Alice and Bob, share many copies of some bipartite state ρ^{AB} in system AB and their goal is to maximize coherence on Bob's subsystem via special operations. The task of assisted coherence distillation via local quantum-incoherent operations and classical communication (LQICC) was firstly introduced in Ref. [16]. Then, Streltsov *et al.* [17] further studied the assisted coherence distillation via the set of operations X , where X is either local incoherent operations and classical communication (LICC), LQICC, separable incoherent operations (SI), or separable quantum-incoherent oper-

ations (SQI). In these sets of operations, Bob is restricted to just incoherent operations on his subsystem, since they do not allow for local creation of coherence on Bob's side. The corresponding distillable coherence on Bob's side is denoted by $C_X^{A|B}$ and can be given as follows [17]:

$$C_X^{A|B}(\rho^{AB}) = \sup\{R : \lim_{n \rightarrow \infty} (\inf_{A \in X} \|Tr_A[\Lambda[\rho^{\otimes n}]] - \Psi_2^{\otimes \lfloor Rn \rfloor}\|) = 0\},$$

where $\lfloor x \rfloor$ is the largest integer below or equal to x and $\Psi_2 = |\Psi_2\rangle\langle\Psi_2|^B$ is a maximally coherent single-qubit state on Bob's side. For any bipartite state ρ^{AB} , the distillable coherence $C_X^{A|B}(\rho^{AB})$ is upper bounded by the QI relative entropy $C_{re}^{A|B}(\rho^{AB}) = \min_{\sigma^{AB} \in \mathcal{QI}} S(\rho^{AB} \| \sigma^{AB})$, where the minimization is taken over the set of all quantum-incoherent states QI [17]. Quite remarkably, both sets of operations SQI and SI lead to the same maximal performance for all states in the task of assisted coherence distillation. Besides, for all pure states and maximally correlated states in the incoherent basis, all the sets of operations we consider are always equivalent in this task and the distillable coherence can reach the upper bound, namely QI relative entropy. However, we do not know whether the sets of operations such as LICC, LQICC and SI (SQI) are equivalent in this task and whether the distillable coherence can reach the upper bound for general mixed states. Therefore, in this paper, we will investigate the assisted coherence distillation of two special classes of mixed states. The results suggest that there exist states, namely Werner states, whose distillable coherence can not reach the upper bound via the set of operations X we consider here.

2 Assisted coherence distillation of states with vanished basis-dependent discord

In the following we discuss the scenario where the state shared by Alice and Bob has vanished basis-dependent discord [23]. In this section, we will frequently refer to a bipartite system AB , and without other stated, we use $\{|i\rangle^A\}$ and $\{|j\rangle^B\}$ as the fixed reference bases of subsystems A and B , respectively. Before we study this work, we recall the condition of states with vanished basis-dependent discord as follows.

Lemma 1. (Yadin *et al.* [23]) Let Π^B be the von Neumann measurement with respect to the fixed reference basis $\{|j\rangle^B\}$ of subsystem B . For any given state ρ^{AB} in system AB , let ρ^A and ρ^B be the reduced states of ρ^{AB} on subsystem A and B , respectively. Then, its basis-dependent discord $D_{\Pi^B}^{A|B}(\rho^{AB})$ is zero, i.e. $D_{\Pi^B}^{A|B}(\rho^{AB}) = S(\rho^{AB} \| \rho^A \otimes \rho^B) - S(\Pi^B(\rho^{AB}) \| \rho^A \otimes \Pi^B(\rho^B)) = 0$, if and only if there exists a decomposition of ρ_{AB} ,

$$\rho^{AB} = \sum_{\alpha} p_{\alpha} \rho_{\alpha}^A \otimes \rho_{\alpha}^B, \quad (1)$$

such that ρ_{α}^B are perfectly distinguishable by projective measurements in the fixed reference basis $\{|j\rangle^B\}$.

It is well known that coherence is a basis-dependent measure of quantumness in single systems, and similarly, $D_{\Pi^B}^{A|B}$ can be seen as a basis-dependent measure of quantumness of correlation [23]. From the Lemma 1, we know that the states with vanished basis-dependent discord are special separable states whose ρ_α^B have disjoint coherence support. Another important result we will refer to is the following Lemma 2 which is about the distillable coherence $C_X^{A|B}$ via the set of operations X we consider here.

Lemma 2. (Streltsov *et al.* [17]) For an arbitrary bipartite state $\rho = \rho^{AB}$ in system AB , the following inequality holds:

$$C_{LICC}^{A|B}(\rho) \leq C_{LQICC}^{A|B}(\rho) \leq C_{SI}^{A|B}(\rho) = C_{SQI}^{A|B}(\rho) \leq C_{re}^{A|B}(\rho), \quad (2)$$

Lemma 2 presents the power of all the sets of operations such as LICC, LQICC, SI and SQI in the task of assisted coherence distillation for an arbitrary bipartite state. Equipped with these results, we are now in position to prove the following theorem.

Theorem 3. For any bipartite state $\rho = \rho^{AB}$ given in Eq.(1), the following equality holds:

$$C_{LICC}^{A|B}(\rho) = C_{LQICC}^{A|B}(\rho) = C_{SI}^{A|B}(\rho) = C_{SQI}^{A|B}(\rho) = C_{re}^{A|B}(\rho) = C_{re}(\rho^B), \quad (3)$$

where ρ^B is the reduced state of ρ^{AB} on subsystem B .

Proof. Note that the QI relative entropy of any bipartite state ρ can be evaluated exactly [16]:

$$C_{re}^{A|B}(\rho) = S(\Pi^B(\rho)) - S(\rho) = D_{\Pi^B}^{A|B}(\rho) + C_{re}(\rho^B). \quad (4)$$

Where Π^B is the von Neumann measurement in the fixed reference basis $\{|j\rangle^B\}$ and $D_{\Pi^B}^{A|B}(\rho)$ is the basis-dependent discord as given in Lemma 1. Combining Eq. (4) and Eq. (2) in Lemma 2, we get the inequality

$$C_{LICC}^{A|B}(\rho) \leq C_{LQICC}^{A|B}(\rho) \leq C_{SI}^{A|B}(\rho) = C_{SQI}^{A|B}(\rho) \leq D_{\Pi^B}^{A|B}(\rho) + C_{re}(\rho^B). \quad (5)$$

Besides, it is straightforward that $C_X^{A|B}(\rho^{AB}) \geq C_{re}(\rho^B)$, where X is either LICC, LQICC, SI or SQI. According to Lemma 1, for any bipartite state $\rho = \rho^{AB}$ given in Eq.(1), its basis-dependent discord $D_{\Pi^B}^{A|B}(\rho)$ is equal to zero. Applying Eq. (5) and combining the aforementioned results we arrive at the desired equality given in Eq. (3). \square

From Theorem 3, we obtain that for all states with vanished (incoherent) basis-dependent discord, there does not exist basis-dependent quantum correlation between Alice and Bob to help Bob improve his distillable coherence. However, we do not know whether the necessity of Theorem 3 is true. In other words, if the distillable coherence $C_X^{A|B}(\rho^{AB})$ via the set of operations X , is equal to the relative entropy of coherence $C_{re}(\rho^B)$, does the state ρ^{AB} has the form given in Eq. (1)? However, as will see in the next section, the distillable coherence of a Werner state can not reach the upper bound and it means that the basis-dependent discord $D_{\Pi^B}^{A|B}(\rho^{AB})$ of a Werner state can not be completely transformed into the coherence on Bob's side via the set of operations X we consider here.

3 Assisted coherence distillation of Werner states

Here, we consider assisted coherence of distillation for the Werner states of the form

$$\rho^{AB} = p|\Phi_+\rangle\langle\Phi_+| + \frac{1-p}{4}I, \quad (6)$$

where $p \in (0, 1)$, and $|\Phi_+\rangle$ is the Bell state denoted as $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (or other Bell states [22]). Here I is the identity operator on the composite system or on the marginal systems, depending on the context. For the two-qubit system, we choose the computable basis as the fixed reference basis of each subsystem. In this case, the reduced state of Bob ρ^B is incoherent and $C_{re}(\rho^B) = 0$. Then, the distillable coherence of Bob via the set of operations X comes from the basis-dependent quantum correlation between Alice and Bob. Moreover, in the following theorem we show that the distillable coherence for a Werner state can not reach the upper bound, namely QI relative entropy and in this case all sets of operations we consider here are equivalent for assisted coherence distillation.

Theorem 4. For any Werner state $\rho = \rho^{AB}$ given in Eq. (6), the following inequality holds:

$$C_{LICC}^{A|B}(\rho) = C_{LQICC}^{A|B}(\rho) = C_{SI}^{A|B}(\rho) = C_{SQI}^{A|B}(\rho) < C_{re}^{A|B}(\rho), \quad (7)$$

Proof. In the first step of the proof we show that for any Werner state ρ^{AB} , its distillable coherence $C_{LQICC}^{A|B}(\rho^{AB})$ is strictly less than its QI relative entropy $C_{re}^{A|B}(\rho^{AB})$. By direct calculations, the QI relative entropy of the Werner state ρ^{AB} is

$$\begin{aligned} C_{re}^{A|B}(\rho^{AB}) &= D_{\Pi_B}^{A|B}(\rho^{AB}) \\ &= \frac{1-p}{4} \log_2(1-p) - \frac{1+p}{2} \log_2(1+p) + \frac{1+3p}{4} \log_2(1+3p). \end{aligned} \quad (8)$$

where $p \in (0, 1)$. Next, let Alice use X basis to measure her subsystem. If the result of Alice's measurement is 1, the post-measurement state of Bob is

$$\rho_+^B = p|+\rangle\langle+| + (1-p)\frac{I}{2}, \quad (9)$$

where $|+\rangle = |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is a maximally coherent single-qubit state, and in this case Bob does not need to do anything. If the result of Alice's measurement is -1 , Alice informs Bob her result and Bob performs the incoherent Z gate $|0\rangle\langle 0| - |1\rangle\langle 1|$ on his subsystem to obtain the state with the form given in Eq. (9). Repeat this process, and Bob obtains many copies of ρ_+^B . Then, the distillable coherence of Bob via this LQICC protocol is

$$C_{re}(\rho_+^B) = \frac{1+p}{2} \log_2(1+p) + \frac{1-p}{2} \log_2(1-p). \quad (10)$$

Consequently, we show the fact that $C_{re}(\rho_+^B)$ is the largest distillable coherence of Bob assisted by Alice through all LQICC protocols. Suppose that

Alice performs a local quantum measurement \mathcal{E} on her subsystem, and informs Bob her results. According to the results of Alice, Bob performs the incoherent unitary operations on his subsystem, and then the state on Bob's side can be denoted as $\rho_{\mathcal{E}}^B = p\tilde{\rho}_{\mathcal{E}}^B + (1-p)\frac{I_B}{2}$. With the convexity of the relative entropy of coherence [4], we only need to consider the case that $\tilde{\rho}_{\mathcal{E}}^B$ is a pure state, which can be written in the Bloch representation,

$$\tilde{\rho}_{\mathcal{E}}^B = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix},$$

where x, y and z are real and satisfy the equation

$$x^2 + y^2 + z^2 = 1. \quad (11)$$

In this case, direct calculations show that

$$C_{re}(\rho_{\mathcal{E}}^B) = \frac{1-p}{2} \log_2(1-p) + \frac{1+p}{2} \log_2(1+p) \\ - \frac{1+pz}{2} \log_2(1+pz) - \frac{1-pz}{2} \log_2(1-pz).$$

Examine the maximum value of $C_{re}(\rho_{\mathcal{E}}^B)$ under the constraint condition Eq. (11), and we obtain the maximum value $C_{re}(\rho_+^B)$ at $x = 1$ and $y = z = 0$. Then, it follows that $C_{re}(\rho_{\mathcal{E}}^B) \leq C_{re}(\rho_+^B)$. Therefore, for the Werner state ρ^{AB} , it holds that $C_{LQICC}^{A|B}(\rho^{AB}) = C_{re}(\rho_+^B)$.

Now, we show that $C_{re}(\rho_+^B)$ is strictly less than the upper bound $C_{re}^{A|B}(\rho^{AB})$. Let Eq. (8) subtract Eq. (10) and the difference is denoted as

$$f(p) = C_{re}^{A|B}(\rho^{AB}) - C_{re}(\rho_+^B) \\ = \frac{1+3p}{4} \log_2(1+3p) - \frac{1-p}{4} \log_2(1-p) - (1+p) \log_2(1+p), \quad (12)$$

with $p \in (0, 1)$. Note that $f(0) = f(1) = 0$, $f(\frac{1}{3}) > 0$ and $f''(p) = \frac{1-3p}{(1+3p)(1-p^2)\ln 2}$. Then, $f(p)$ is strictly convex for all $p \in (0, \frac{1}{3})$ and strictly concave for all $p \in (\frac{1}{3}, 1)$. Besides, using $f'(p) = 0$, we get the minimum value at $p = 0$. Therefore, we obtain that $f(p) > 0$ for all $p \in (0, 1)$ and plot the function $f(p)$ in Fig. 1 and Fig. 2, which highlights the distillable coherence $C_{re}(\rho_+^B)$ via the set of operations X is strictly less than the upper bound, namely QI relative entropy for any Werner state given in Eq. (6).

Finally, we show that for the Werner state ρ_{AB} , the distillable coherence of Bob is also equal to $C_{re}(\rho_+^B)$ via other sets of operations such as SQI, SI and LICC. Since LQICC is a subset of SQI and SI, the similar approach as above leads to the equality: $C_{SI}^{A|B}(\rho) = C_{SQI}^{A|B}(\rho) = C_{re}(\rho_+^B)$. In order to prove that $C_{LICC}^{A|B}(\rho) = C_{re}(\rho_+^B)$, we present a LICC protocol achieving the rate $C_{re}(\rho_+^B)$. Let Alice perform an erasing (incoherent) measurement on her subsystem [24], whose Kraus operators are

$$K_1^A = \begin{pmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, K_2^A = \begin{pmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix},$$

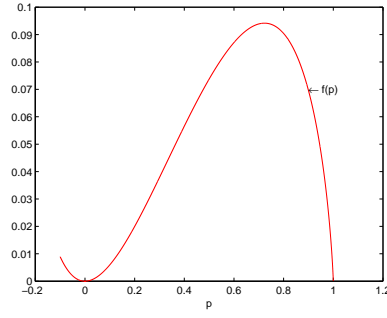


Fig. 1 Graph of function $f(p)$ which is the difference between the QI relative entropy and the distillable coherence via LQICC for any Werner state given in Eq. (6).

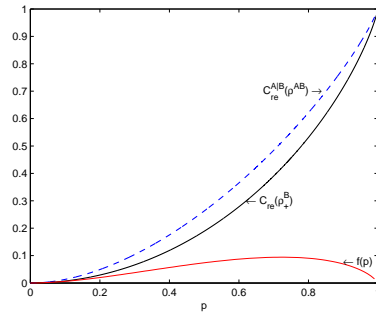


Fig. 2 Distillable coherence of Bob is strictly less than the upper bound for any Werner state ρ^{AB} given in Eq. (6) in the task of assisted coherence distillation. We plot the distillable coherence $C_{re}(\rho_+^B)$ (the black line), the upper bound $C_{re}^{A|B}(\rho^{AB})$ (the dashed line), and the function $f(p)$ given in Eq. (12) (the red line).

with $(K_1^A)^\dagger K_1^A + (K_2^A)^\dagger K_2^A = I$. If the result of Alice's measurement is 1, Bob's post-measurement state is $\rho_1^B = p(\frac{i|0\rangle + |1\rangle}{\sqrt{2}})(\frac{-i\langle 0| + \langle 1|}{\sqrt{2}}) + (1-p)\frac{I}{2}$. In this case, Bob can get the state ρ_+^B by applying first an X gate $|1\rangle\langle 0| + |0\rangle\langle 1|$ and then an incoherent unitary gate $|0\rangle\langle 0| - i|1\rangle\langle 1|$ to ρ_1^B . If the result of Alice's measurement is 2, Bob's post-measurement state is $\rho_2^B = p(\frac{-i|0\rangle + |1\rangle}{\sqrt{2}})(\frac{i\langle 0| + \langle 1|}{\sqrt{2}}) + (1-p)\frac{I}{2}$, and then Bob can get ρ_+^B by applying first an X gate $|1\rangle\langle 0| + |0\rangle\langle 1|$ and then a phase gate $|0\rangle\langle 0| + i|1\rangle\langle 1|$ to ρ_2^B . Therefore, via LICC, the distillable coherence of Bob is $C_{re}(\rho_+^B)$ with the similar arguments as above. These results imply the inequality given in Eq. (7). \square

Note that for standard coherence distillation the relative entropy coherence is in fact equal to the optimal distillation rate [15]. In the task of assisted coherence distillation via the set of operations X , the QI relative entropy $C_{re}^{A|B}$ is an upper bound on $C_X^{A|B}$ [16, 17]. However, by Theorem 4, we know that the distillation coherence $C_X^{A|B}$ is not always able to reach the upper bound, namely QI relative entropy. It is surprising that for a Werner state

ρ^{AB} given in Eq. (6) the distillable coherence $C_X^{A|B}(\rho^{AB})$ does not depend on the particular set of operations performed by Alice and Bob. In particular, in this case the optimal distillation rate can already be obtained by the weakest set of operations LICC. The other sets of operations, such as LQICC, SQI, and SI, can not provide any advantage to improve the distillable coherence on Bob's side. Through the above results, we strongly conjecture that all the sets of operations we consider here are equivalent in the task of assisted coherence distillation for any mixed state and we leave this question open for further study.

4 Conclusions

In this paper, we have discussed the assisted coherence distillation of some mixed states, such as the states with vanished (incoherent) basis-dependent discord and Werner states, via the set of operations X , where X is either LICC, LQICC, SI or SQI. For these mixed states, all sets of operations we consider here are equivalent for assisted coherence distillation. In particular, we have provided a sufficient condition such that Alice's assistance via the set of operations X can not help Bob improve his distillable coherence, and the condition is that the state shared by Alice and Bob has vanished basis-dependent discord. Moreover, we have proved that the distillable coherence of a Werner state via the set of operations X can not reach the upper bound, namely QI relative entropy. This result suggest that there exist a state for example the Werner state whose distillable coherence is not its QI relative entropy, even through the largest set of operations SQI. We hope that our work helps to well understand the resource theory of coherence in distributed scenarios.

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